8.1 Integration by Parts

Consider $\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}$.
We can write this as $u(x)\frac{dv(x)}{dx} = \frac{d}{dx}(u(x)v(x)) - v(x)\frac{du(x)}{dx}$.
If we integrate both sides, we obtain $\int u(x)\frac{dv(x)}{dx}dx = \int \frac{d}{dx}(u(x)v(x))dx - \int v(x)\frac{du(x)}{dx}dx$ or $\int u(x)dv = u(x)v(x) - \int v(x)du$.
If a and b are in the interval where $u(x)$ and $v(x)$ are continuos, and the integrals are defined from a to b, $\int_a^b u(x)dv = u(x)v(x)|_a^b - \int_a^b v(x)du$.
eg 1 $\int xcos(x)dx; \quad \int x^2 ln(x)dx, \quad \int xe^x dx, \quad \int xsin(x^2)dx \quad \int tan^{-1}(x)dx, \ \int ln(x)dx;$ $\int (ln(x))^2 dx; \ \int e^x sin(x)dx$
eg 2 Find the volume of revolution about the x-axis of the area in the first quadrant of $y = ln(x)$ and $y = 1$.
eg 3 Find the volume of revolution about the x-axis of the area in the first quadrant of $y = ln(x)$ and $x = 1$.

8.2 Trigonometric Integrals

A. Integrals of the form $sin^m(x)\,cos^n(x)$
a. If m and n are even, use the half angle formulas $sin^2(x) = \frac{1-cos(2x)}{2}$, $cos^2(x) = \frac{1+cos(2x)}{2}$.

eg 4 Find the volume generated when $y = cos(x)$ is rotated about the x-axis in [0,π/2].

By the Disks method, $v = 4\pi\int_0^{\pi/2} cos^2(x)dx = 2\pi\int_0^{\pi} 1 + cos(2x)dx = 2\pi\left(x + \frac{sin(2x)}{2}\right)|_0^{\pi/2} = \pi^2$ $cu.\ units$

b. If m is odd and n even, save a sine and make the rest of the factors cosines by using a Pythagorean identity.

eg 5 $\int sin^3(x)cos^2(x)dx = \int (1 - cos^2(x))cos^2(x)sin(x)dx = \frac{cos^5(x)}{5} - \frac{cos^3(x)}{3} + c$

c. If m is even and n odd, save a cosine and make the rest of the factors sine by using a Pythagorean identity.

eg 6 $\int \frac{cos^3(x)}{\sqrt{sin(x)}}dx = \int \frac{(1-sin^2(x))}{\sqrt{sin(x)}}cos(x)dx = 2\sqrt{sin(x)} - \frac{2sin^{5/2}(x)}{5} + c =$
$\frac{2}{5}\sqrt{sin(x)}(5 - sin^2(x)) + c$

B. Integrals of the form $\sec^m(x)\tan^n(x)$

$\boldsymbol{a}$. If m is even, save a $sec^2(x)$ and make the rest tangent by using a Pythagorean identity.
eg 7 $\int tan^5(x)sec^4(x)dx = \int tan^5(x)\left(tan^2(x)+1\right)sec^2(x)dx = \frac{tan^8(x)}{8} + \frac{tan^6(x)}{6} + c$
eg 8 $\int sec^4(x)dx = \int\left(tan^2(x)+1\right)sec^2(x)dx = \frac{tan^3(x)}{3} + tan(x) + c$
$\boldsymbol{b}$. If n is odd, save a $sec(x)tan(x)$ and make the rest secant by using a Pythagorean identity.
eg 9 $\int tan^3(x)sec^6(x)dx = \int\left(sec^2(x)-1\right)sec^5(x)\,sec(x)tan(x)dx = \frac{tan^8(x)}{8} + \frac{tan^6(x)}{6} + c$
$\boldsymbol{c}$. If $m=0$ and n even, save a $tan^2(x)$ and change the rest to secant by using a Pythagorean identity.
eg 10 $\int tan^2(x)dx = \int\left(sec^2(x)-1\right)dx = tan(x) + x + c$
eg 11 Find the area bounded by $y = tan^4(x)$ and the x-axis between $x=0$ and $x=\pi/4$.
$\int_0^{\pi/4} tan^4(x)dx = \int_0^{\pi/4}\left(sec^2(x)-1\right)tan^2(x)dx = \int_0^{\pi/4} sec^2(x)tan^2(x) - tan^2(x)dx =$
$\int_0^{\pi/4} sec^2(x)tan^2(x) - \left(sec^2(x)-1\right)dx = \left(\frac{tan^3(x)}{3} - tan(x) + x\right)\Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{2}{3}$
$\boldsymbol{d}$. If $n=0$ and m odd, do integration by parts.
eg 12 $\int sec^3(x)dx = sec(x)tan(x) - \int sec(x)tan^2(x)dx =$
$sec(x)tan(x) - \int sec(x)\left(sec^2(x)-1\right)dx =$
$sec(x)tan(x) - \int sec^3(x)dx + \int sec(x)dx.$
Since $\int sec^3(x)dx = sec(x)tan(x) - \int sec^3(x)dx + ln|sec(x)+tan(x)|$, by solving for $\int sec^3(x)dx$ we obtain $\int sec^3(x)dx = \frac{1}{2}(sec(x)tan(x) + ln|sec(x)+tan(x)|)$

C. Integrals of the form $\csc^m(x)\cot^n(x)$
$\boldsymbol{a}$. If m is even, save a $\csc^2(x)$ and make the rest cotangent by using a Pythagorean identity.
eg 13 $\int cot^4(x)csc^4(x)dx = \int cot^4(x)\left(cot^2(x)+1\right)csc^2(x)dx = \frac{-cot^7(x)}{7} - \frac{cot^5(x)}{5} + c$
$\boldsymbol{b}$. If n is odd, save a $csc(x)cot(x)$ and make the rest cosecant by using a Pythagorean identity.
eg 14 $\int cot^3(x)csc^6(x)dx = \int\left(csc^2(x)-1\right)csc^5(x)\,csc(x)cot(x)dx = \frac{-csc^8(x)}{8} - \frac{cot^6(x)}{6} + c$

Practice:
1) $\int tan^4(x)dx = \frac{tan^3x}{3} - tanx - x + c$
2) $\int tan^3(x)\,sec^3(x)dx = \frac{sec^5x}{5} - \frac{sec^3x}{3} + c$
3) $\int tan^2(x)\,sec^4(x)dx = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + c$
4) $\int tan^2(x)\,sec(x)dx = \frac{1}{2}sec\,x\,tan\,x - \frac{1}{2}ln\,|sec\,x + tan\,x| + c$
5) $\int cot^3(x)\,csc^3(x)dx = -\frac{csc^5x}{5} + \frac{csc^3x}{3} + c$
6) $\int csc(x)dx = -ln\,|csc\,x + cot\,x| + c$
7) $\int cot^3(x)\,csc^4(x)dx = -\frac{cot^6x}{6} - \frac{csc^4x}{4} + c$
8) $\int cot^3(x)\,dx = -\frac{cot^2x}{2} - ln|sin\,x| + c$

8.3 Trigonometric Substitutions

A) Integrals odd powers of $\sqrt{a^2 - x^2}$, use $x = a\,sin(\theta)$ as substitution.

eg 15 Find $\int \frac{\sqrt{1-x^2}}{x^2} dx$
If we use the substitution $x = sin(\theta)$ we find
$\int \frac{\sqrt{1-x^2}}{x^2} dx = \int \frac{cos(\theta)}{sin^2(\theta)} cos(\theta) d\theta = \int cot^2(\theta) d\theta = \int \left(csc^2(\theta) - 1\right) d\theta =$
$-\,cot(\theta) - \theta = \; - \; \frac{\sqrt{1-x^2}}{x} - sin^{-1}(x) + c$ after making a triangle with angle θ.

B) Integrals odd powers of $\sqrt{a^2 + x^2}$, use $x = a\,tan(\theta)$ as substitution.

eg 16 Find $\int \frac{dx}{x^2\sqrt{1+x^2}}$
If we use the substitution $x = \tan(\theta)$ we find
$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{sec^2(\theta)}{tan^2(\theta) sec(\theta)} d\theta = \int \frac{cos(\theta)}{sin^2(\theta)} d\theta = \; - csc(\theta) = \; - \; \frac{\sqrt{x^2+1}}{x} + c,$ after making a triangle with angle θ.

C) Integrals odd powers of $\sqrt{x^2 - a^2}$, use $x = a\,sec(\theta)$ as substitution.

eg 17 Find $\int \frac{dx}{\sqrt{x^2-4}}$
If we use the substitution $x = 2sec(\theta)$ we find
$\int \frac{dx}{\sqrt{x^2-4}} = \int 2\, \frac{sec(\theta)tan(\theta)}{2tan(\theta)} d\theta = \int sec(\theta) d\theta = ln|sec(\theta) + tan(\theta)| = ln|\frac{x+\sqrt{x-4}}{2}| + c,$ after making a triangle with angle θ.

eg 18 $\int \frac{x^2}{\sqrt{4-x^2}} dx; \; \int \frac{\sqrt{x^2-9}}{x} dx;$

eg 19 Find the volume of revolution about the x-axis of the area in the first quadrant of $y = \frac{1}{x^2+1}$ and the line $x = \pi/4$.

8.4 Integration by Partial Fractions
Case I Distinct Linear Factors ($Heaviside$ "$Cover\text{-}up$" $Method$)

eg 20 Find $\int \frac{x+1}{x(x-1)(x-2)}dx$.

By partial fractions $\frac{x+1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-2)}$.

By the cover-up method $A = \frac{x+1}{(x-1)(x-2)}|_0 = \frac{1}{2}$; $B = \frac{x+1}{x(x-2)}|_1 = -2$; $C = \frac{x+1}{x(x-1)}|_2 = \frac{3}{2}$ or $\frac{x+1}{x(x-1)(x-2)} = \frac{1}{2x} - \frac{2}{(x-1)} + \frac{3}{2(x-2)}$.

So $\int \frac{x+1}{x(x-1)(x-2)}dx = \int \frac{1}{2x} - \frac{2}{(x-1)} + \frac{3}{2(x-2)}dx = ln|\frac{\sqrt{x}(x-2)^{3/2}}{(x-1)^2}| + c$.

eg 21 Find $\int \frac{x^3}{(x-1)(x+1)}dx$
Since $\frac{x^3}{(x-1)(x+1)} = x + \frac{x}{(x-1)(x+1)}$ by long division, and $\frac{x}{(x-1)(x+1)} = \frac{1/2}{(x-1)} + \frac{1/2}{(x+1)}$ by the Cover-up method, $\int \frac{x^3}{(x-1)(x+1)}dx = \int x + \frac{1/2}{(x-1)} + \frac{1/2}{(x+1)}dx = \frac{x^2}{2} + ln\sqrt{x^2-1} + c$

Case II Repeated Factors

eg 22: Find $\int \frac{x+1}{x(x-1)^3}dx$.

By partial fractions $\frac{x+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}$ where
$A = \frac{x+1}{(x-1)^3}|_0 = -1$; $B = \frac{x+1}{x}|_1 = 2$; $C = \frac{\left(\frac{x+1}{x}\right)'}{1!}|_1 = -1$; $D = \frac{\left(\frac{x+1}{x}\right)''}{2!}|_1 = 1$
so $\frac{x+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)}$, or $\int \frac{x+1}{x(x-1)^3}dx =$
$\int -\frac{1}{x} + \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)}dx = ln|\frac{x-1}{x}| - \frac{1}{(x-1)^2} + \frac{1}{(x-1)} + c$

eg 23 Find $\int \frac{x+2}{x^4-2x^3+x^2}dx$
By partial fractions $\frac{x+2}{x^4-2x^3+x^2} = \frac{x+2}{x^2(x-1)^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}$ where
$A = \frac{x+2}{(x-1)^2}|_0 = 2$; $C = \frac{x+2}{x^2}|_1 = 3$; $B = \frac{\left(\frac{x+2}{(x-1)^2}\right)'}{1!}|_0 = 5$; $D = \frac{\left(\frac{x+2}{x^2}\right)'}{2!}|_1 = -5$
or $\frac{x+2}{x^2(x-1)^2} = \frac{2}{x^2} + \frac{5}{x} + \frac{3}{(x-1)^2} - \frac{5}{(x-1)}$, so $\int \frac{x+2}{x^2(x-1)^2}dx =$
$\int \frac{2}{x^2} + \frac{5}{x} + \frac{3}{(x-1)^2} - \frac{5}{(x-1)}dx = ln|\frac{x}{x-1}|^5 - \frac{2}{x} - \frac{3}{(x-1)} + c$

eg 24 Find $\int \frac{x^3-x}{x^4-2x^2}dx$
This integral can be found using partial fractions, but if we look closely we can see that if $u = x^4 - 2x^2$, $du = 4\left(x^3 - x\right)dx$, so $\int \frac{x^3-x}{x^4-2x^2}dx = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}ln|x^4 - 2x^2| + c$

Case III Irreducible Quadratic Factors

eg 25 Find $\int \frac{x+1}{x(x^2+1)}dx$.

Since $\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$, by cover-up on the first term we obtain

$A = \frac{x+1}{(x^2+1)}|_0 = 1$. By cover-up on the second term and evaluated at either root $\pm i$ we obtain $\frac{x+1}{x} \mid_{-i} = Bx + C|_{-i} \Rightarrow 1 + i = B(-i) + C \Rightarrow 1 + i = C - Bi$

If we equate real and imaginary parts, we obtain $C = 1, B = -1$. Since the partial fraction expansion of $\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{-x+1}{(x^2+1)}$, $\int \frac{x+1}{x(x^2+1)}dx = \int \frac{1}{x} + \frac{-x+1}{(x^2+1)}dx = ln\frac{|x|}{\sqrt{x^2+1}} + tan^{-1}(x) + c$.

eg 26 Find $\int \frac{x+1}{x(x^2+1)}dx$ by using systems of equations.

Since $\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$, by cover-up $A = 1$ so $\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{Bx+C}{(x^2+1)}$.

If we multiply by the LCD of all the fractions, we obtain $x + 1 = x^2 + 1 + x(Bx + C)$ or $x = x^2 + x(Bx + C)$.

If we evaluate the expression at any number, say $x = 1$, we obtain $B = -C$.
If we evaluate the same expression at another number, say $x = -1$, we obtain $-B + C = 2$.
If we solve the system, we have $C = 1$ and $B = -1$. Since the partial fraction expansion of $\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{-x+1}{(x^2+1)}$, $\int \frac{x+1}{x(x^2+1)}dx = \int \frac{1}{x} + \frac{-x+1}{(x^2+1)}dx = ln\frac{|x|}{\sqrt{x^2+1}} + tan^{-1}(x) + c$.

eg 27. Find $\int \frac{x}{(x^2+1)(x^2+9)}dx$.
Since $\frac{x}{(x^2+1)(x^2+9)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+9)}$, by cover-up on the first term we obtain

By cover-up on the first term evaluating at either root $\pm i$, we obtain $\frac{x}{(x^2+9)} \mid_i = Bx + C|_i$ $\Rightarrow \frac{1}{8}i = Bi + C$. If we equate real and imaginary parts, we obtain, $B = \frac{1}{8}$, and $C = 0$.
By cover-up on the second term evaluating at either root $\pm 3i$, we obtain $\frac{x}{(x^2+1)} \mid_{3i} = Bx + C|_{3i} \Rightarrow -\frac{3}{8}i = B\,3i + C$. If we equate real and imaginary parts, we obtain, $B = -\frac{1}{8}$, and $C = 0$.

Since the partial fraction expansion of $\frac{x}{(x^2+1)(x^2+9)} = \frac{1/8}{(x^2+1)} + \frac{-1/8}{(x^2+9)}$,
$\int \frac{x}{(x^2+1)(x^2+9)}dx = \frac{1}{8}\int \frac{1}{(x^2+1)} - \frac{1}{(x^2+9)}dx = \frac{1}{8}ln\left(\frac{x^2+1}{x^2-9}\right) + c$

Basic Integration Procedures

1) Basic Substitutions

$a.\int \frac{x+1}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2+2x+3}+c$, by the substitution $u = x^2+2x+3$.

$b.\int \frac{x^2-2x}{x^3-3x^2+1}dx = \frac{1}{3}ln|x^3-3x^2+1|+c$, by the substitution $u = x^3-3x^2+1$.

$c.\int \frac{2}{1+(2x-3)^2}dx = tan^{-1}(2x-3)+c$

Try $\int \frac{1}{e^x+e^{-x}}dx$ ANS: $tan^{-1}(e^x)+c$

2) Completing the Square

$a.\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-16+8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} = sin^{-1}\left(\frac{x-4}{4}\right)+c$

$b.\int \frac{2}{x^2-6x+13}dx = \int \frac{2}{(x-3)^2+2^2}dx = tan^{-1}\left(\frac{x-3}{2}\right)+c$

$c.\int \frac{dt}{\sqrt{2t-t^2}} = \int \frac{dt}{\sqrt{1-(t-1)^2}} = sin^{-1}(t-1)+c$

Try $\int \frac{dt}{\sqrt{-t^2+4t-3}}$ ANS: $sin^{-1}(t-2)+c$

3) Trigonometric Identities

$a.\int tan(x)csc(x)dx = \int sec(x)dx = ln|sec(x)+tan(x)|+c$

$b.\int csc^2(x)sin(2x)dx = \int 2cot(x)dx = ln|sin(x)|^2+c$

$c.\int (sec(x)+tan(x))^2dx = \int \left(sec^2(x)+2sec(x)tan(x)+tan^2(x)\right)dx =$
$\int \left(sec^2(x)+2sec(x)tan(x)+[sec^2(x)-1]\right)dx = 2tan(x)+2sec(x)-x+c$

Try $\int (sec(x)+cot(x))^2dx$ ANS: $tan(x) = 2ln|csc(x)-cot(x)|-cot(x)-x+c$

4) Reducing Improper Fractions

$a.\int \frac{3x^2-7x}{3x+2}dx = \int \left(x-3+\frac{6}{3x+2}\right)dx = \frac{x^2}{2}-3x+2ln|3x+2|+c$

$b.\int \frac{x^2}{x^2+1}dx = x-tan^{-1}(x)+c$

$c.\int \frac{2x^3}{x^2-1}dx = \int 2x+\frac{2x}{x^2-1}dx = x^2+ln|x^2-1|+c$

Try $\int \frac{4x^3-x^2+16x}{x^2+4}dx$ ANS: $2x^2-x+2tan^{-1}\left(\frac{x}{2}\right)+c$

5) Separating Fractions

$a.\int \frac{1-x}{\sqrt{1-x^2}}dx = \int \left(\frac{1}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\right)dx = sin^{-1}(x)+\sqrt{1-x^2}+c$

$b.\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}dx = \int \left(\frac{1}{2\sqrt{x-1}}+\frac{1}{x}\right)dx = \sqrt{x-1}+ln(x)+c$

$c.\int_0^{\pi/4} \frac{1+sin(x)}{cos^2(x)}dx = tan(x)+sec(x)\Big|_0^{\pi/4} = \sqrt{2}$

Try $\int_0^{1/2} \frac{2-8x}{1+4x^2}dx$ ANS: $\frac{\pi}{4}-ln(2)$

6) Substitutions

An integral containing a term $(ax+b)^{p/q}$, let $u=(ax+b)^{1/q}$

$a. \int \frac{dx}{x\sqrt{1+x}} \rightarrow$ let $u=(1+x)^{1/2}$, $u^2=1+x,\ 2udu=dx$ so

$\int \frac{dx}{x\sqrt{1+x}} = \int \frac{2}{u^2-1}du = \ ln|\frac{u-1}{u+1}| + c = ln|\frac{\sqrt{1+x}\ -1}{\sqrt{1+x}\ +1}| + c$

$b. \int \frac{dx}{x^{1/2}-x^{1/4}} \rightarrow$ let $u=x^{1/4}$, $u^4=x,\ 4u^3du=dx;$ so $\int \frac{dx}{x^{1/2}-x^{1/4}} = 4\int \frac{u^2}{u-1}du$

$= 4\int u+1\frac{1}{u-1}du = 4\left(\frac{u^2}{2}+u+ln|u-1|\right)+c = 2x^{1/2}+4x^{1/4}+4ln|x^{1/4}-1|+c$

$c. \int \frac{1}{x(x+1)^{\frac{3}{2}}}dx \ =$ let $u=(x+1)^{1/2}$, $u^2=x+1,\ 2udu=dx$ so $\int \frac{1}{x(x+1)^{\frac{3}{2}}}dx$

$= \int \frac{2}{(u^2-1)u^2}du = \int \frac{1}{u-1} - \frac{1}{u+1} - \frac{2}{u^2}du = \ ln|\frac{u-1}{u+1}| + \frac{2}{u} + c = ln|\frac{\sqrt{x+1}\ -1}{\sqrt{x+1}\ +1}| + \frac{2}{\sqrt{x+1}} + c$

Try $\int \frac{1}{\sqrt{x}(x-1)}dx$ ANS: $ln|\frac{\sqrt{x}\ -1}{\sqrt{x}\ +1}| + c$

8.8 Improper Integrals

The definition of the Riemann integral $\int_a^b f(x)dx$ as a finite limit of a Riemann sum can only be used if the interval of integration is finite and the function is bounded on that interval. Improper integrals have unbounded functions in interval of integration or infinite intervals of integration.

Improper integral can Converge, Diverge to $\pm\infty$ or Diverge without limit.

eg 28 a) Improper integrals of unbounded functions

$\int_2^5 \frac{1}{2\sqrt{x-2}}dx$ $\int_0^3 \frac{1}{x-1}dx$ $\int_0^1 ln(x)dx$ $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

eg 29 b) Improper integrals with infinite intervals of integration

$\int_1^\infty \frac{1}{x}dx$ diverges; $\int_1^\infty \frac{1}{x^2}dx$ converges to 1; $\int_1^\infty \frac{1}{x^p}dx$ converges to $\frac{1}{p-1}$ for $p>1$;

$\int_0^\infty sin(x)dx$ diverges without limit; $\int_{-\infty}^\infty \frac{1}{x^2+1}dx$ converges to π; $\int_0^\infty \frac{1}{x^2}dx$ diverges to ∞;

$\int_{-1}^1 \frac{1}{x}dx =$ diverges without a limit; $\int_{-1}^1 \frac{1}{x^2}dx\ =$ diverges to ∞; $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$ converges to $\pi/2$.

Find the area bounded by $f(x)=\frac{2}{x^2-1}$ in [3,∞]; area bounded by $f(x)=\frac{x}{\sqrt{1-x^2}}$ in [0,1].

Comparison Test for Improper Integrals.

Let $f(x)$ and $g(x)$ be continous functions such that $f(x) > g(x)$ for $x \geq a$.

If $\int_a^\infty f(x)dx$ converges, $\int_a^\infty g(x)dx$ converges. If $\int_a^\infty g(x)dx$ diverges, $\int_a^\infty f(x)dx$ diverges.

eg 30 $\int_1^\infty \frac{2+e^{-x}}{x} dx$ diverges since , $\frac{2+e^{-x}}{x} > \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\, dx$ diverges. $(p = 1)$

eg 31 $\int_1^\infty \frac{x}{\sqrt{1+x^6}} dx$ converges since , $\frac{x}{\sqrt{1+x^6}} < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\, dx$ converges. $(p > 1)$

eg 32 $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$ converges since , $\frac{e^{-x}}{\sqrt{x}} < \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges.